2.12 Introduction to Acceleration due to gravity :

All bodies, irrespective of their mass or nature, falling freely in vacuum, will have the same acceleration at a given place. This acceleration is called acceleration due to gravity.

It is denoted by 'g' and its value differs from place to place. It is greatest at the poles and the least at the equator. It is generally taken to be 981 cm/s^2 in CGS system and 9.81 m/s^2 in MKS system on the surface of the Earth. It is determined with the help of a simple pendulum or a compound pendulum.

2.13 The Simple Pendulum :

A simple pendulum consists of a heavy spherical bob (ideally may be treated as point mass) suspended from a fixed point by an inextensible, weightless string.

In Figure 2.6, S is the point of suspension, O is the centre of the bob, 'x' is the displacement of the bob from O and 'l' is the length of the pendulum. Let OS is the position of bob at rest and SA is the displaced position of the bob through angle 'θ'.

The pendulum in A position is subjected to two forces (i) its weight acting vertically downward and (ii) the tension of the string along AS towards its point of suspension.

The force mg due to its weight can be resolved into two components (i) mgcosθ acting along SA and (ii) mgsinθ at right angles to SA.

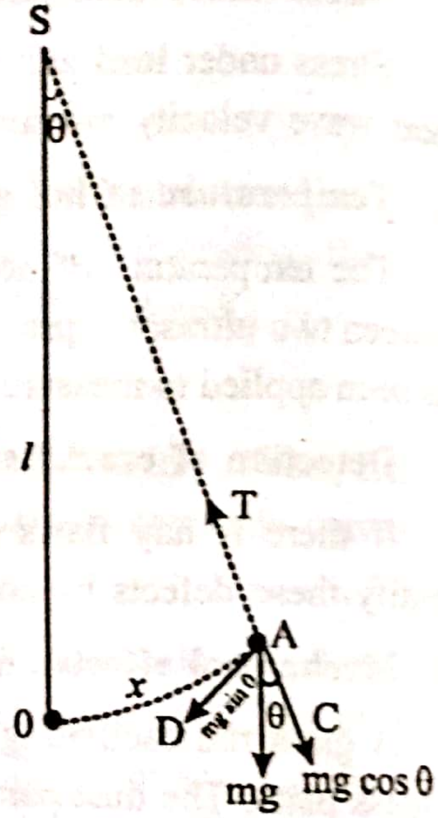

Fig. 2.6

As there is no motion along SA the force mgcosθ balances the tension in the string. Hence the only force which acts on the bob is mgsin θ and acts towards the mean position of the bob O.

$$\text{Acceleration of the bob} = \frac{mg\sin\theta}{m} = g\sin\theta. \quad [\because \text{acceleration} = \text{force / mass}]$$

This acceleration is directed toward AO or towards the mean position of the bob 0. If θ is very small we have $\sin\theta \approx \theta$

$\therefore$ Acceleration of the bob = g θ But $\theta = \dfrac{\text{arc}}{\text{radius}} = \dfrac{OA}{SO} = \dfrac{x}{l}$

$\therefore$ Acceleration of the bob $= \frac{gx}{l}$ ... ... (2.4)

Thus, the bob's acceleration is proportional to its displacement from its mean position (x) and always directed towards its mean position i. e. it executes simple harmonic motion. The periodic time (T) of the pendulum is

$$\therefore \quad T = 2\pi \sqrt{\frac{1}{g/l}} = 2\pi \sqrt{\frac{l}{g}} \qquad \text{... ... . (2.5)}$$

$$\therefore \quad g = 4\pi^2 \left(\frac{l}{T^2}\right) \qquad \text{... ... . (2.6)}$$

Hence, by substituting periodic time (T) and length of the simple pendulum (l) in above equation we can determine 'g'.

➤ **Drawbacks of Simple Pendulum :**

This method of finding the value of g has the following drawbacks :

- Simple pendulum is only ideal concept because point mass bob and weightless string is practically not possible.
- We have not considered effect of the air resistance and buoyancy.
- The expression for the time period is true only if displacement is infinitely small.
- The motion of the bob is not a motion of translation. The rotary motion of the bob should also consider in the formula.
- The bob is not rigidly connected with the string and has some relative motion with respect to thread when approaching the limits of the string.

2.14 Compound OR Physical Pendulum :

Definition : A rigid body capable for oscillating in a vertical plane about a fixed horizontal axis through pivot is called compound or physical pendulum.

The motion of compound pendulum is simple harmonic and its periodic time is given by

$$T = 2\pi \sqrt{\frac{I}{mgl}} \qquad \text{... ... . (2}$$

Where 'I' is the moment of inertia of a rigid body about the axis through point of suspension and 'm' is the mass of rigid body, 'l' be the pendulum length or the distance between the point of suspension and centre of gravity.

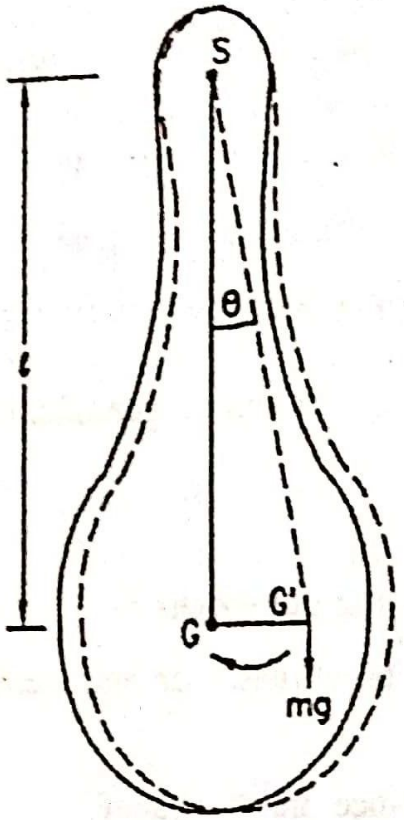

Fig. 2.7

In Fig. 2.7, S is the point of suspension of a compound pendulum. The center of gravity of a rigid body lies vertically below the point of suspension in equilibrium position. Let θ be the angle between rest and deflected position. The couple acting on a rigid body is given by

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$= \vec{l} \times m\vec{g} \quad (\because r = l)$$

$$= mgl \sin\theta$$

Due to this couple the rigid body will tends to come its original position.

If ω be the angular velocity of the pendulum then $\frac{d\omega}{dt}$ is the angular accleration. The torque or couple acting on a pendulum is $I\frac{d\omega}{dt}$

$$\therefore \quad \vec{\tau} = I\vec{\alpha} = I\frac{d\omega}{dt} \qquad \ldots\ \ldots\ \ldots. \ (2.8)$$

But $\quad \vec{\tau} = \vec{r} \times \vec{F}$

$$= \vec{l} \times m\vec{g}$$

$$= -mgl\sin\theta \qquad \ldots\ \ldots\ \ldots. \ (2.9)$$

Negative sign indicates the restoring couple.

Now, comparing equations (2.8) and (2.9) we get,

$$I\frac{d\omega}{dt} = -mgl\sin\theta$$

If the amplitude of oscillation is small then $\sin\theta = \theta$

$$\therefore \quad I\frac{d\omega}{dt} = -mgl\,\theta$$

$$\frac{d\omega}{dt} = -\frac{mgl}{I}\theta$$

$$= -\mu\theta \qquad \left(\text{Taking } \frac{mgl}{I} = \mu\right)$$

$$\therefore \quad \frac{d\omega}{dt} + \mu\theta = 0 \qquad \text{......... (2.10)}$$

This is the equation of motion of compound pendulum. Equation (2.10) shows that angular acceleration is proportional to the angular displacement. This angular acceleration executes simple harmonic motion.

The periodic time is given by

$$T = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{\frac{1}{mgl/I}}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{I}{mgl}} \qquad \text{......... (2.11)}$$

If I_o be the moment of inertia of rigid body about the axis through center of gravity and parallel to the point of suspension.

Then by using parallel axis theorem

$$I = I_o + ml^2 \qquad \text{......... (2.12)}$$

The effective distance of the particle of a body from its axis of rotation is called **radius of gyration** and its value depends upon the distribution of mass in the body and the position and direction of the axis of rotation.

The radius of gyration can be defined as the perpendicular distance from the axis of rotation and the point where the whole mass of the body were to be concentrated. Its expression is given by

$$k = \sqrt{\frac{r_1^2 + r_2^2 + \ldots + r_N^2}{N}}$$

Hence the radius of gyration k is the root mean square distance of the particle from the axis of rotation.

If k is the radius of gyration then $I_0 = mk^2$.

Substituting this value in equation (2.12) we get ,

$$I = mk^2 + ml^2 \quad \text{... ... ... (2.13)}$$

From equations (2.11) and (2.13)

$$T = 2\pi\sqrt{\frac{mk^2 + ml^2}{mgl}}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{k^2/l + l}{g}} \quad \text{... ... ... (2.14)}$$

The period of oscillation is same as the simple pendulum of length $\frac{k^2}{l} + l$ or $\frac{k^2 + l^2}{l}$. This length is called **the length of equivalent simple pendulum or reduced length of pendulum.** It is denoted by L. Since k^2 is never zero. Hence equivalent simple pendulum length is always greater than l.

2.15 Centre of Oscillation :

Let O be the point on the other side of the centre of gravity at a distance $\frac{k^2}{l}$. This point is called centre of oscillation. The axis passing through this point and parallel to the point of suspension is called axis of oscillation.

In Fig. 2.8,

$$SO = SG + GO$$

$$= l + \frac{k^2}{l}$$

$$\therefore \quad L = l + l' \qquad (\because l' = k^2 / l)$$

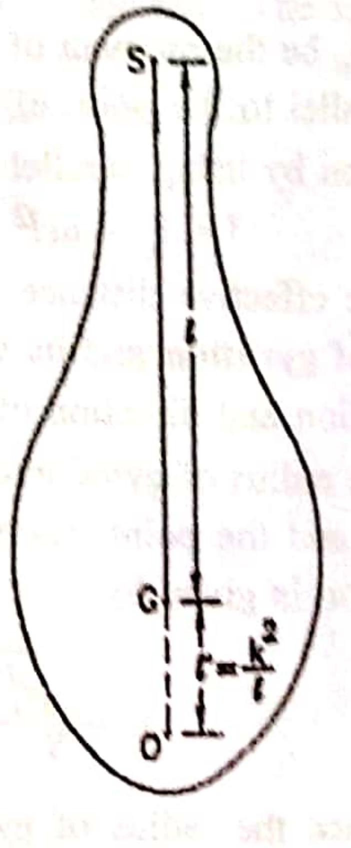

Fig. 2.8

The periodic time is

$$T = 2\pi\sqrt{\frac{k^2/l + l}{g}} = 2\pi\sqrt{\frac{L}{g}} \quad \ldots \quad \ldots \quad \ldots (2.15)$$

Thus the point of oscillation O lies vertically at a distance L from the point of suspension. The distance between these two points is called the length of equivalent simple pendulum.

2.16 Interchangeability of Centers of Suspension and Oscillation :

If the pendulum be inverted and suspended about the axis of oscillation through O, as shown in Figure 2.9. Its time period will be obviously given by $T = 2\pi\sqrt{\frac{k^2 + l'^2}{gl'}}$.

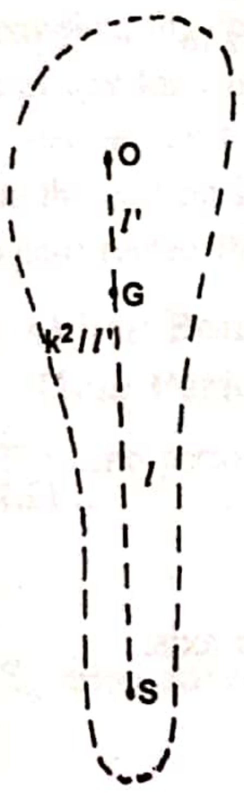

Fig. 2.9

Since $\frac{k^2}{l} = l'$ we have $k^2 = l\,l'$ so that the expression for the time period t becomes

$$T = 2\pi\sqrt{\frac{ll'/l + l}{g}} \quad \ldots \quad \ldots \quad \ldots (2.16)$$

$$\therefore \quad T = 2\pi\sqrt{\frac{l' + l}{g}} = 2\pi\sqrt{\frac{L}{g}} \quad \ldots \quad \ldots \quad \ldots (2.17)$$

The time period about point of oscillation is same as the axis of suspension through S. Thus the centre of oscillation and oscillation are interchangeable.

2.17 Centre of Percussion :

Fig. 2.10 shows a section of a rigid body of mass 'm' by a vertical plane passing though its centre of gravity and point of suspension 'S'.

Let an impulsive force $\vec{F}$ be applied at O in a direction perpendicular to the vertical line passing thiough SGO and the axis of suspension through S. Then this force is equivalent to (i) an equal and a like parallel force $\vec{F}$ at G and (ii) a clockwise couple, formed by the force $\vec{F}$ at O and an equal and opposite force $\vec{F}$ at G, the moment of which is equal to $\vec{F} \times \vec{l'}$, Where l' = distance GO.

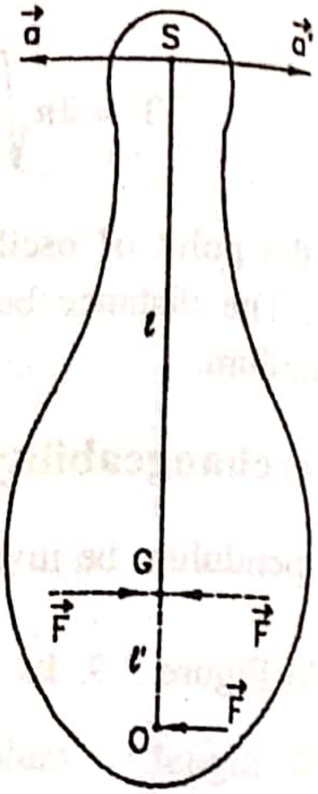

Fig. 2.10

This force $\vec{F}$ at G tends to produce linear acceleration in the body i.e. $\vec{F} = m\vec{a}$. The acceleration $\vec{a} = \frac{\vec{F}}{m}$ is in the direction of force i.e., from right to left.

This is the acceleration produced at S by the force at G. The couple on the other hand tends to produce an angualr accelration $\vec{\alpha}$ in the body about a parallel axis through G. If I be the moment of inertia about the axis through centre of gravity then

$$I\vec{\alpha} = \vec{F} \times \vec{l'}$$

$$\therefore \quad \vec{\alpha} = \frac{\vec{F} \times \vec{l'}}{I}$$

$$\therefore \quad \vec{\alpha} = \frac{\vec{F} \times \vec{l'}}{mk^2} \qquad \text{... (2.18)}$$

where $I = mk^2$, k is radius of gyration.

But, linear acceleration = angular acceleration × distance from the axis.

The linear acceleration is produced by this couple is given by

$$\vec{a'} = \vec{l} \times \vec{\alpha}$$

$$= \vec{l} \times \vec{F} \times \frac{\vec{l'}}{mk^2} \qquad \text{... ... ... (2.19)}$$

The direction of $\vec{a'}$ is from left to right i.e. in opposite to that of $\vec{a}$.

Hence, force $\vec{F}$ applied at O may produce no effect at S.

i.e. $\vec{a} = \vec{a}'$

$$\therefore \frac{\vec{F}}{m} = \vec{l} \times \vec{F} \times \frac{\vec{l}'}{mk^2}$$

$$\therefore l' = \frac{k^2}{l} \quad \dots \quad \dots \quad \dots (2.20)$$

This is the distance of the point O from the CG of the body. The point O is the centre of oscillation. This point is called the centre of percussion with respect to S.

Thus if a body be struck at the centre of percussion, or the centre of oscillation, in a direction perpendicular to its axis of suspension, it does not move body as a whole, about the point of suspension, but simply turns about the axis passing through it.

This explains when a ball strikes against a bat at a point of oscillation or the cenre of percussion, the corresponding point in the hand as the point of suspension, no sting or shock of any kind is felt. The bat being held in the hands of the player has really no - well - defined axis about which is turns, but the player instinctively knows where to receive the ball on it so that no sting or shock is felt. Similarly, a good hammer should be so constructed that its centre of percussion lies in a line with the driving force.

2.18 Other Points, Collinear with Centre of Gravity about which the Time Period is the same :

The time period for compound pendulum is

$$t = 2\pi\sqrt{\frac{k^2 + l^2}{gl}} \quad \dots \quad \dots \quad \dots (2.21)$$

Squaring above equation, we get

$$t^2 = 4\pi^2\left(\frac{l^2 + k^2}{lg}\right)$$

$$\therefore \quad lgt^2 = 4\pi^2 (l^2 + k^2)$$

$$\therefore \quad \frac{lg}{4\pi^2} t^2 = l^2 + k^2$$

$$\therefore \quad l^2 - \frac{gt^2}{4\pi^2} l + k^2 = 0 \quad \dots \quad \dots \quad \dots (2.22)$$

This is a quadratic equation in l.

We know that, the solution of quadratic equation $ax^2 + bx + c = 0$ ... ... (2.

is $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Comparing the coefficients of equation (2.22) and (2.23) we have

$$a = l, \quad b = \frac{-gt^2}{4\pi^2} \quad \text{and} \quad c = k^2$$

The solution of equation (2.22) is given by

$$l = \frac{\frac{gt^2}{4\pi^2} \pm \sqrt{\frac{g^2t^4}{16\pi^4} - 4k^2}}{2} \quad \text{... ... ... (2.2}$$

'which has two solutions

$$l_1 = \frac{gt^2}{8\pi^2} + \sqrt{\frac{g^2t^4}{64\pi^4} - k^2} \quad \text{... ... ... (2.2}$$

and $$l_2 = \frac{gt^2}{8\pi^2} - \sqrt{\frac{g^2t^4}{64\pi^4} - k^2} \quad \text{... ... ... (2.}$$

Adding equations (2.25) and (2.26), we get

$$l_1 + l_2 = \frac{2gt^2}{8\pi^2}$$

$$l_1 + l_2 = \frac{gt^2}{4\pi^2} \quad \text{... ... ... (2}$$

Now, multiplying equation (2.25) and (2.26) we get,

$$l_1 l_2 = \left(\frac{gt^2}{8\pi^2}\right)^2 - \left(\frac{g^2t^4}{64\pi^4} - k^2\right)$$

$$\therefore \quad l_1 l_2 = k^2 \quad \text{... ... ... (2}$$

Equations (2.27) and (2.28) show that l_1 and l_2 are positive. Thus, there are two points of the compound pendulum for the same periodic time at a distance l_1 and $\frac{k^2}{l_1}$ from the centre of gravity.

If we draw a circle of radius l_1 and $\frac{k^2}{l_1} = l_2$ about the centre of gravity, we get four points on the axis passing through point of suspension and centre of gravity. The time period is same for these four points. Hence these points on the axis are said to be collinear about the centre of gravity which has same periodic time.

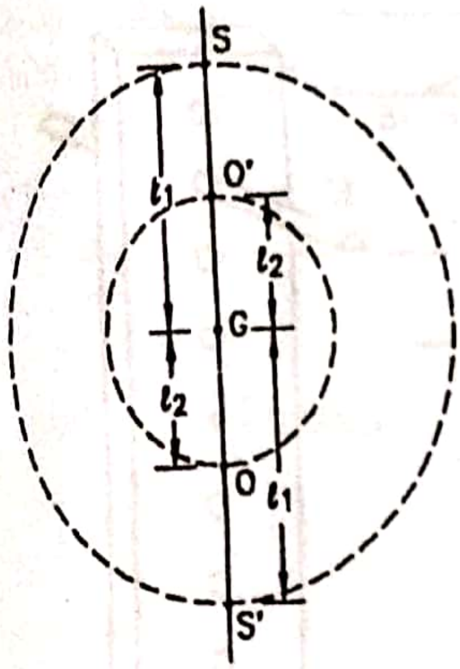

Fig. 2.11

In fig. 2.11 O is the centre of oscillation for S and O' is centre of oscillation for S'.

$$\therefore \ SO = S'O' = l_1 + l_2 = l_1 + \frac{k^2}{l_1} = L \quad \ldots \ldots (2.29)$$

From equation (2.27) and (2.29) we can write

$$l_1 + l_2 = L = \frac{gt^2}{4\pi^2}$$

$$\therefore \quad g = \frac{4\pi^2 L}{t^2} \quad \ldots \quad \ldots \quad \ldots (2.30)$$

Using equation (2.30) we can determine the value of g.

2.19 Conditions for Maximum and Minimum Time Period :

We know that,

$$t = 2\pi\sqrt{\frac{l^2 + k^2}{lg}}$$

$$\therefore \quad t^2 = 4\pi^2\left(\frac{l^2 + k^2}{lg}\right)$$

$$\therefore \quad t^2 = \frac{4\pi^2}{g}\left(\frac{k^2}{l} + l\right)$$

Differentiating above equation with respect to l, we get

$$2t\frac{dt}{dl} = \frac{4\pi^2}{g}\left(-\frac{k^2}{l^2}+1\right) \qquad \cdots \quad \cdots \quad \cdots (2.31)$$

- The time period t will be minimum when $\frac{dt}{dl} = 0$, for $k^2 = l^2$ or $l = k$. Hence if the distance between point of suspension and centre of gravity is equal to the radius of gyration, t will be minimum.
- When $l = 0$, t will be maximum i.e. when the axis passing through the centre of gravity, the periodic time is maximum.

2.20 Bar Pendulum :

A uniform metal bar having holes drilled along its length symmetrically on both the side of its centre of gravity which is suspended from horizontal knife - edge and oscillate in a vertical plane is called bar pendulum. Fig. 2.12 shows a bar endulum.

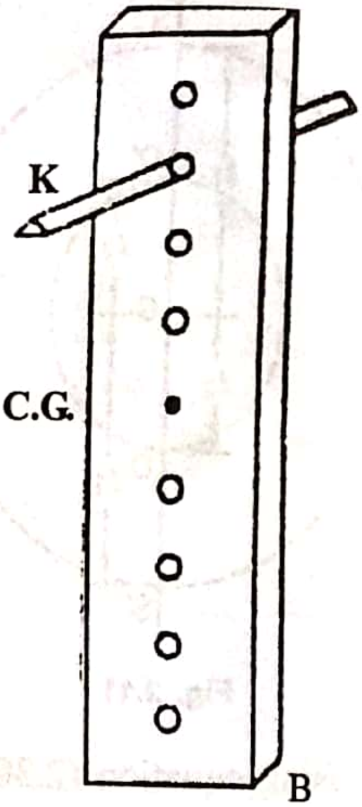

Fig. 2.12

A bar pendulum is a particular case of a compound pendulum. The time period is determined by fixing the knife edge in each hole. The distance of each hole from the centre of gravity is measured. A graph is drawn between the distance from the cenre of gravity along the x-axis and the corresponding time period along the y-axis. The graph is as shown in fig. 2.13.

Fig. 2.13 shows that, the time period decrease at first, acquires a minimum value and then increases as the centre of gravity of the bar is approached, finally becoming infinite at the cenre of gravity.

Now, if a horizontal line JN be drawn, it cuts the two curves at points J, K, M and N about which the time period is same. Therefore, JM = KN = L is the length of equivalent simple pendulum. The points J and M (or K and N) lie on opposite sides of the centre of gravity. They corresponds to the points of suspension and oscillation of the pendulum respectively. The distance between them thus giving directly the length of the equivalent simple pendulum L. The time period.

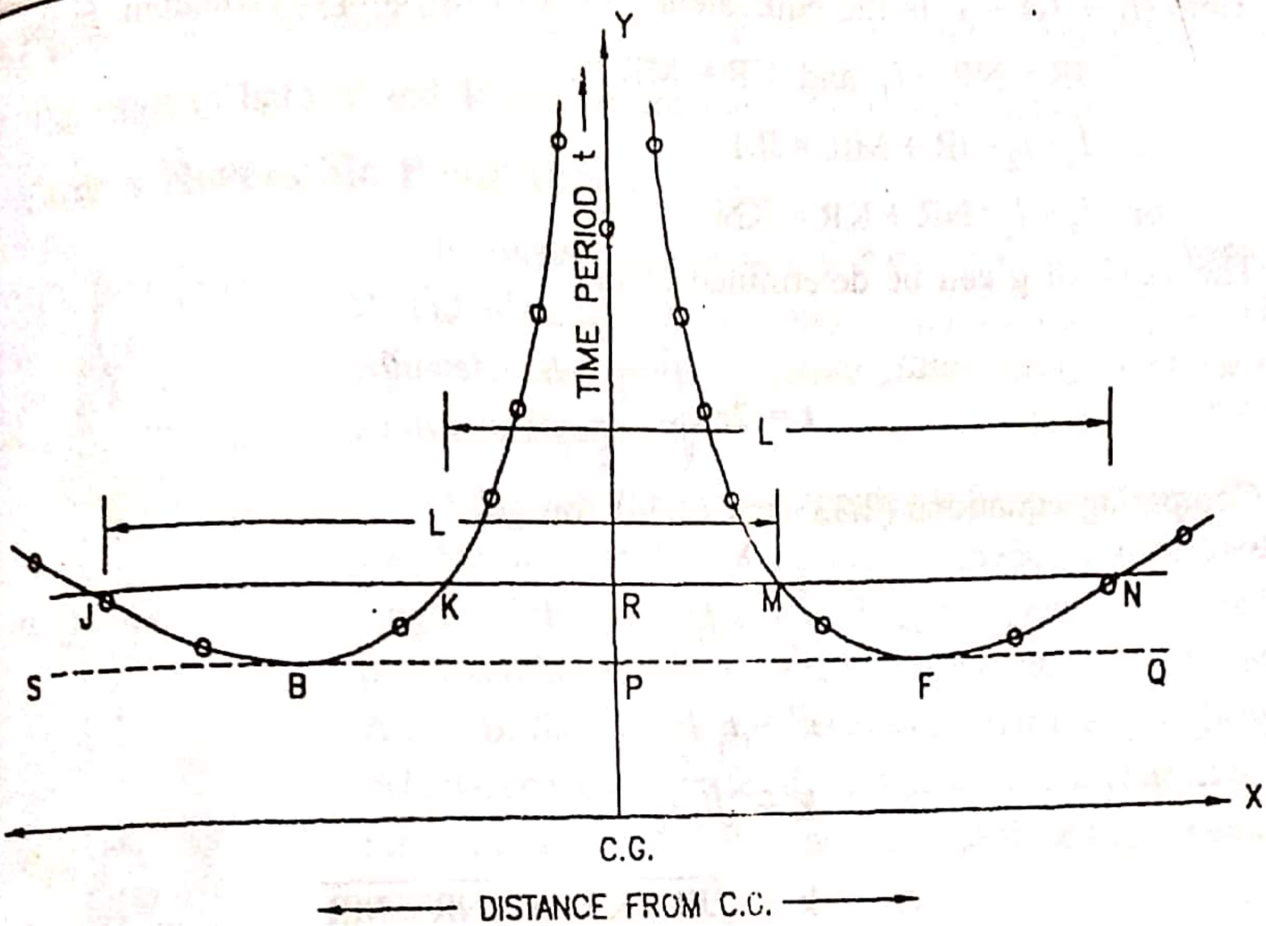

Fig. 2.13

$$t = 2\pi\sqrt{\frac{L}{g}}$$

$$\therefore \quad g = \frac{4\pi^2 L}{t^2} \qquad \dots \quad \dots \quad \dots (2.32)$$

Hence, knowing L and t, we can determine the value of acceleration due to gravity 'g'.

- **Determination of k :**

If we draw the tangential line SQ, touching the two curves at the points B and F respectively. Then at B and F the centre of suspension and oscillation coincide with each other with the condition of minimum time period.

We know that

$$t = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{l_1 + l_2}{g}} \qquad \dots \quad \dots \quad \dots (2.33)$$

Here $(l_1 + l_2) = L$ is the equivalent length of the simple pendulum. In the graph

$$JR = NR = l_1 \text{ and } KR = MR = l_2,$$

$$\therefore \quad l_1 + l_2 = JR + MR = JM$$

$$\text{or} \quad l_1 + l_2 = NR + KR = KN$$

The value of g can be determined from

$$t = 2\pi\sqrt{\frac{k^2/l_1 + l_1}{g}} \quad \dots \quad \dots \quad \dots (2.34)$$

Comparing equations (2.33) and (2.34), we get

$$\frac{k^2}{l_1} = l_2$$

$$\therefore \quad k^2 = l_1 l_2$$

$$\therefore \quad k = \sqrt{l_1 l_2} \quad \dots \quad \dots \quad \dots (2.35)$$

$$\therefore \quad k = \sqrt{JR \times KR} = \sqrt{NR \times MR} \quad \dots \quad \dots \quad \dots (2.36)$$

Using equation (2.36) the radius of gyration k of the bar pendulum can be easily determined.

- The alternate method for accurate determination of radius of gyration 'k' was suggested by Ferguson in 1928.

We know that relation

$$l^2 + k^2 = \frac{lt^2}{4\pi^2} g$$

$$l^2 = \left(\frac{g}{4\pi^2}\right) lt^2 - k^2 \quad \dots \dots (2.37)$$

This is the equation of straight line.

Hence we plot the graph between lt^2 along the x-axis and l^2 along the y-axis gives a straight line as shown in fig. 2.14.

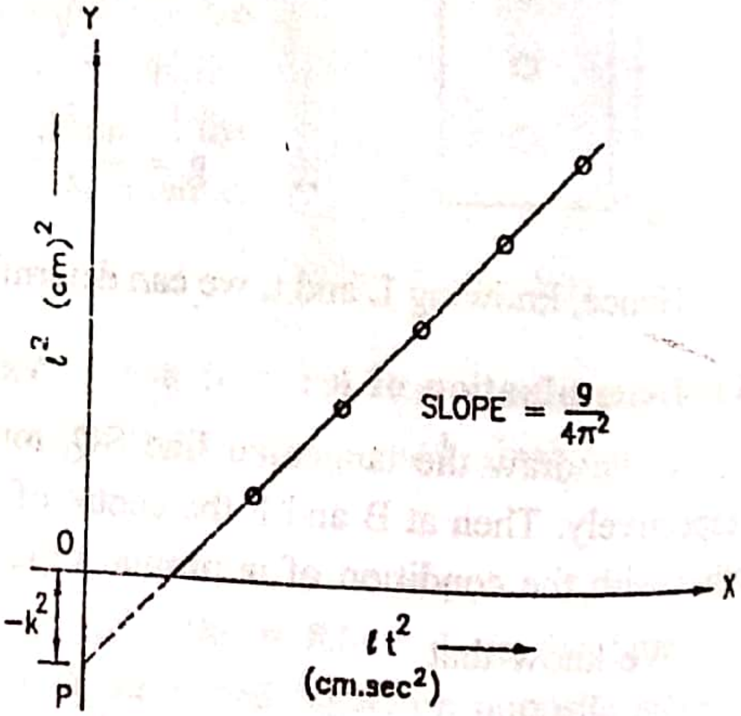

Fig. 2.14

The slope of the line must be equal to $\frac{g}{4\pi^2}$ and its intercept on the y-axis is equal to $-k^2$. The value of both 'g' and 'k' can be obtained much more accurately.

2.21 Kater's Reversible Pendulum :

Fig. 2.15

It consists of a long rod AB having a fixed heavy bob B and also two fixed knife edge F_1 and F_2 which are adjustable and mutually facing at the two ends of the rod as shown in figure 2.15.

The pendulum can be suspended from either side. Two weights W_1 and W_2 which can be made to slide along the length of the bar are clamped in the position desired. The position of the CG is changed by the adjustment of the weights A and B, their position is so chosen that the CG always lies in between the two knife edges. The weight W_2 has an attached micrometer screw arrangement for its finer adjustment of its position.

The fact that centre of suspension and oscillation can be interchanged and it is used in determining the equivalent simple pendulum length, thus g, from a Kater's pendulum. The time period of a pendulum about a horizontal axis at a distance l_1 from its centre of gravity T_1 is given by

$$T_1 = 2\pi\sqrt{\frac{k^2 + l_1^2}{l_1 g}} \qquad \text{... ... ... (2.38)}$$

$$\therefore \quad l_1 T_1^2 = \frac{4\pi^2}{g}(k^2 + l_1^2) \qquad \text{... ... ... (2.39)}$$

Now if we reverse the pendulum and make it to vibrate about another parallel horizontal axis a distance l_2 from its centre of gravity, the time period T_2 is given by

$$T_2 = 2\pi\sqrt{\frac{k^2 + l_2^2}{l_2 g}} \qquad \text{... ... ... (2.40)}$$

$$\therefore \quad l_2 T_2^2 = \frac{4\pi^2}{g}\left(k^2 + l_2^2\right) \qquad \text{... ... ... (2.41)}$$

If two axes are such that they are reciprocal axes of suspension and oscillation then $T_1=T_2$.

Let us substitute $T_1= T_2 = T$ in equations (2.39) and (2.41), we have

$$(l_1 - l_2)\,T^2 = \frac{4\pi^2}{g}(l_1^2 - l_2^2) \text{ if } (l_1 \neq l_2).$$

$$\therefore \quad T^2 = \frac{4\pi^2}{g}(l_1 + l_2)$$

$$\therefore \quad T^2 = \frac{4\pi^2}{g}L \qquad \text{... ... ... (2.42)}$$

Thus the distance between the two points on opposite sides of the CG and collinear with it but situated at unequal distances from it gives the length of the equivalent simple pendulum (L). Once the value of L is known the g can be calculated from the relation as T can be determined i.e. $g = 4\pi^2\left(\frac{L}{T^2}\right)$.

Solved Numerical

Example 2.1 : A piezoelectric X-cut crystal plate has a thickness of 1.6 mm. If the velocity of propagation of sound waves along the X-direction is 5760 m/s, calculate the fundamental frequency of the crystal.

Solution :

Here, Thickness = 1.6 mm = 1.6×10^{-3} m

Velocity = 5760 m/s

In the lowest mode of vibration, the distance between the two faces of the crystal of thickness t will be $\lambda/2$.

$$\therefore \quad t = \frac{\lambda}{2}$$

i.e. $\lambda = 2t = 2 \times 1.6 \times 10^{-3} = 3.2 \times 10^{-3}$ m

$$f = \frac{v}{\lambda} = \frac{5760}{3.2\times10^{-3}} = 1800 \times 10^3 \text{ Hz} = 1.8 \text{ MHz}$$

$\therefore$ Fundamental frequency = 1.8 MHz